Introduction to Computational Fluid Dynamics with Gerris

Stéphane Popinet et al.

NIWA, d'Alembert etc...

Outline

1. Applied mathematics and numerical methods

2. Gerris' strengths: adaptivity and user interface

3. The Navier–Stokes equations

   Plateau–Rayleigh instability, atomisation, complex boundaries

4. The Saint-Venant equations

   Tsunamis

(Classical) mathematical physics in two equations

- System of conservation laws ("transport" processes)

$$\partial_t \int_{\Omega} \mathbf{Q} \, dv + \int_{\partial\Omega} \mathbf{F}(\mathbf{Q}) \cdot \mathbf{n} \, ds = \mathbf{S}$$

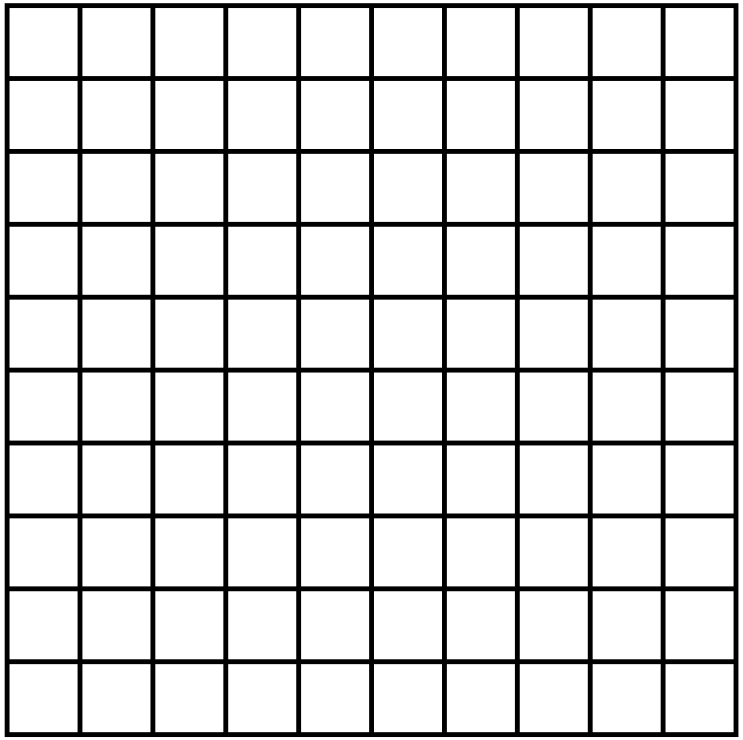
- Helmholtz equation ("diffusive" processes)

$$\partial_t \int_{\Omega} \mathbf{Q} + \int_{\partial\Omega} \alpha \, \nabla\mathbf{Q} \cdot \mathbf{n} \, ds = \mathbf{S}$$

Describes: compressible and incompressible fluids, advection/diffusion, waves (acoustic, gravity, electromagnetic), electromagnetism, elastic solid mechanics etc...

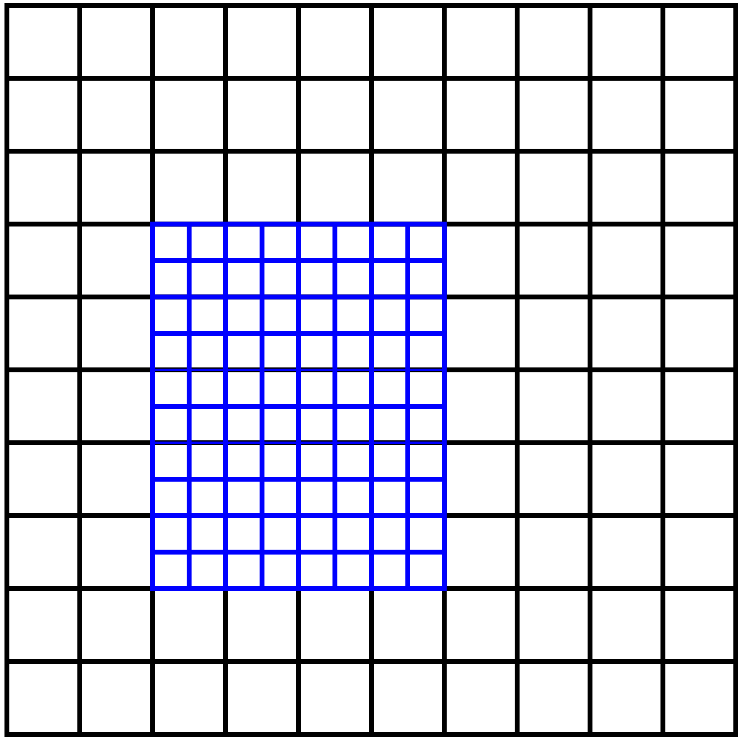
Simple spatial discretisation techniques

Cartesian



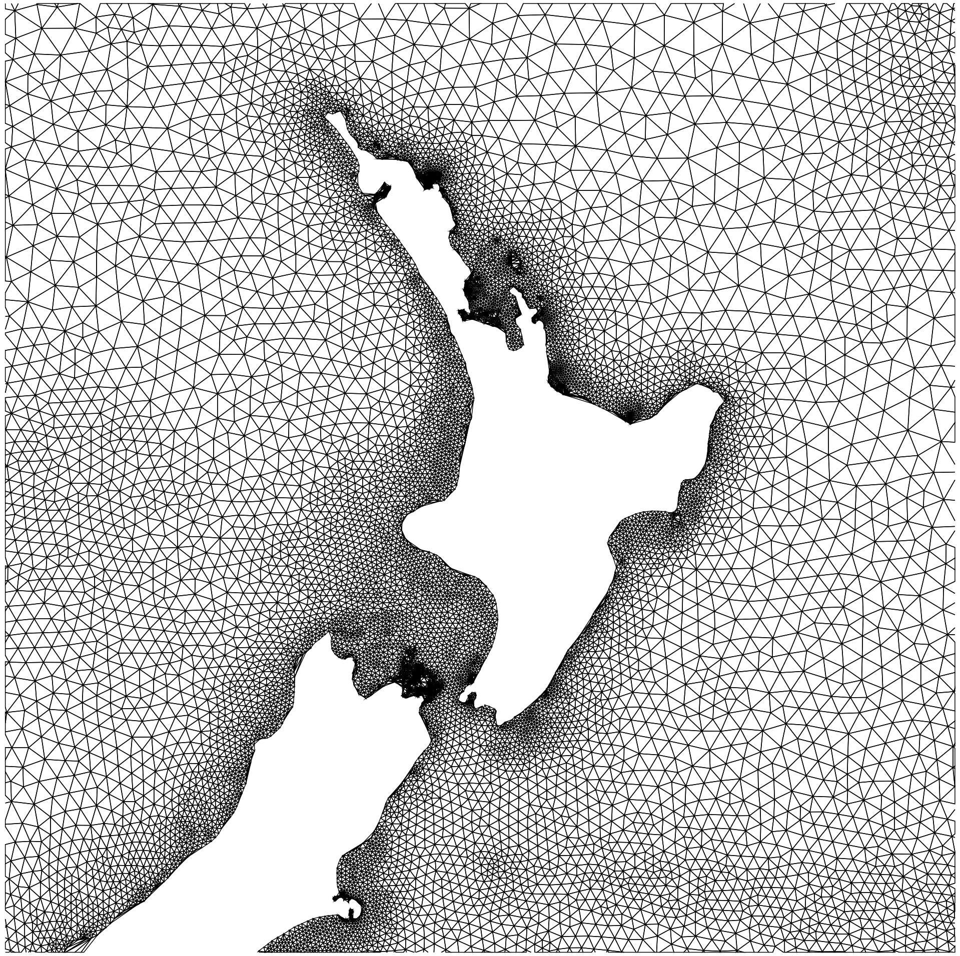
Not adaptive

Nested Cartesian



Some spatial adaptivity

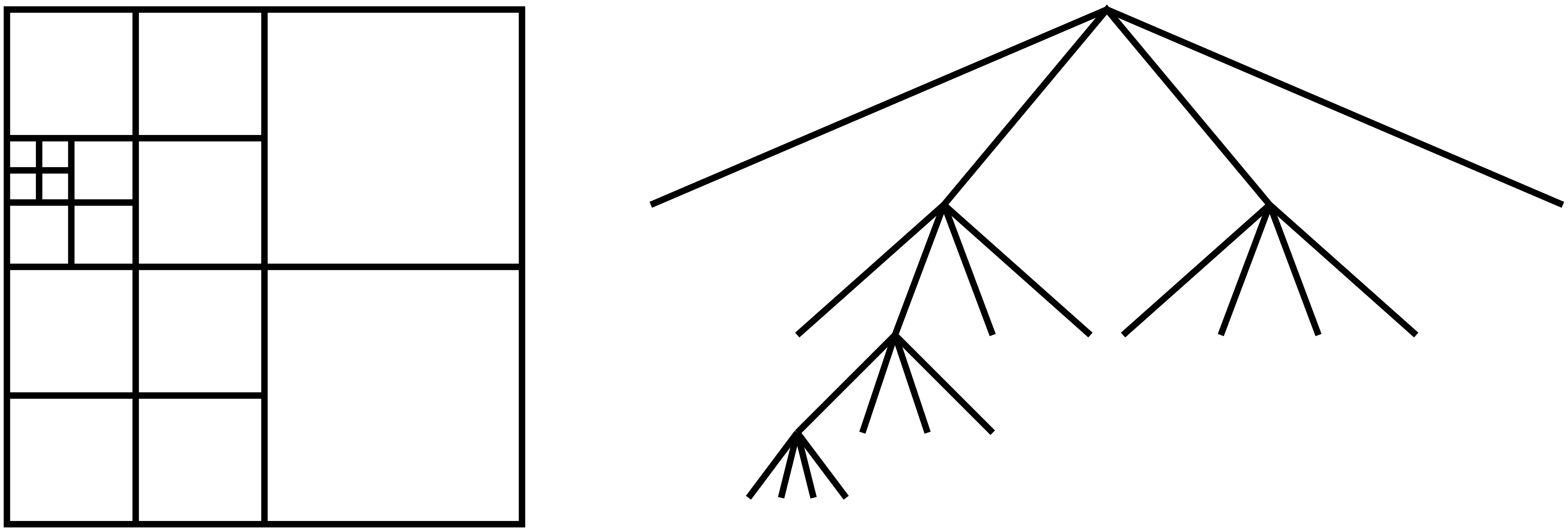
Unstructured statically refined mesh



Fully adaptive in space

Walters and Goring

Dynamic refinement using quadtrees



Fully adaptive in space and time

Robust and automatic meshing

“The curse of dimensionality”

or is adaptive mesh refinement necessary?

- The universe has (at least) four dimensions

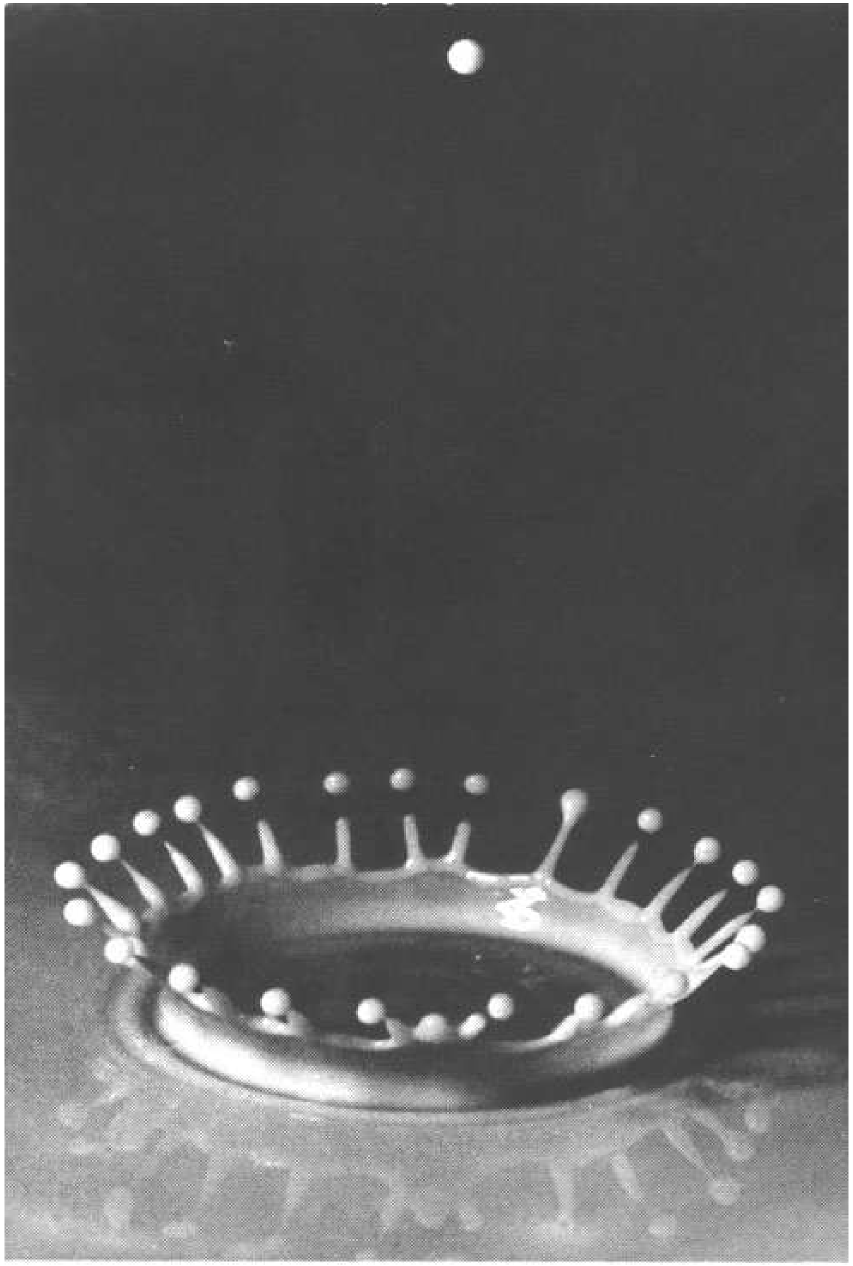
- Using regular Cartesian grids, solution costs scale like

$$C\Delta^{-4}$$

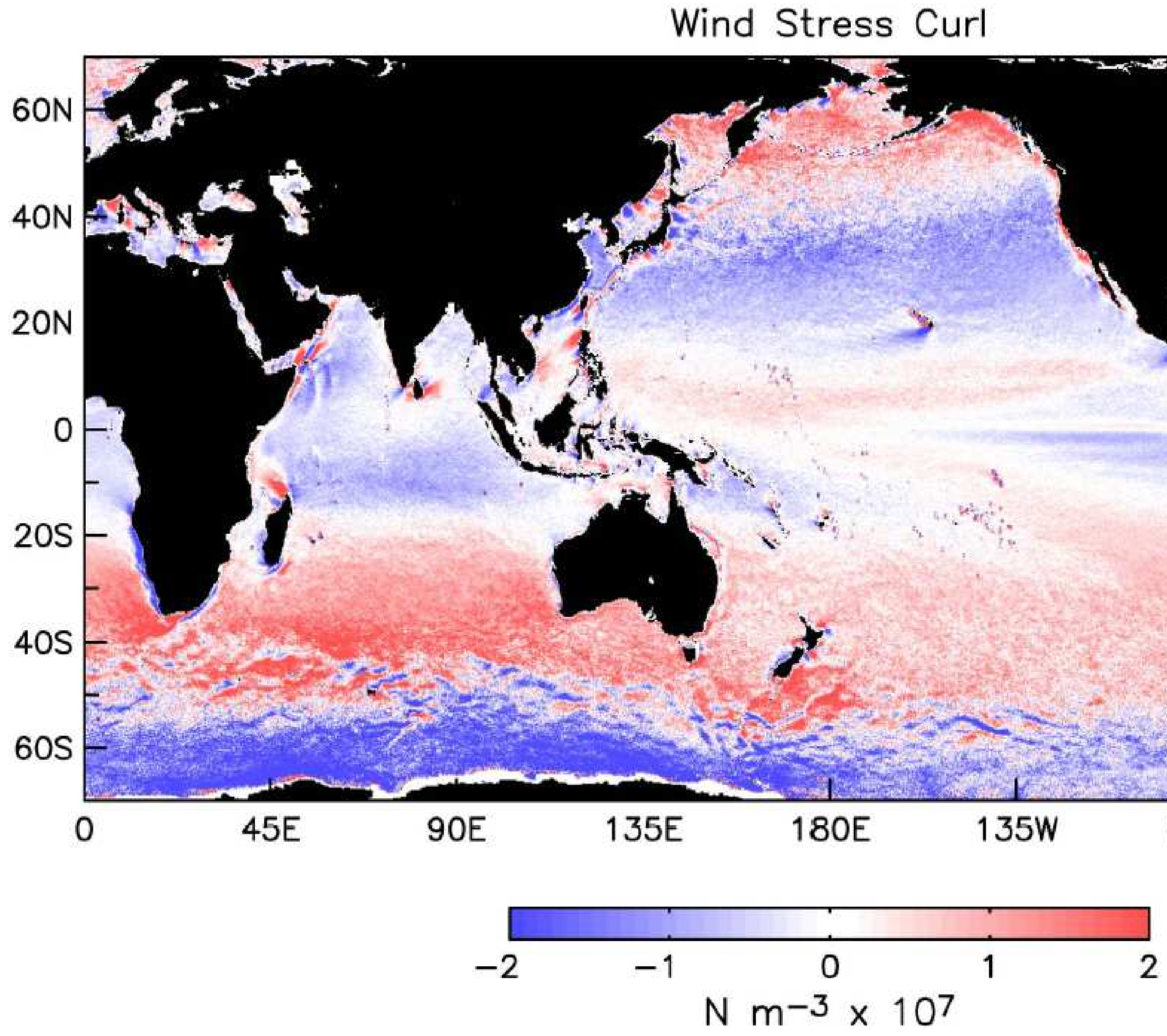
  with C a constant and Δ the spatial resolution in each dimension

- Just buy bigger computers! A 100-fold increase in computing power will buy you a $\sqrt[4]{100} \approx 3$-fold increase in resolution... (assuming C does not increase)

- Can adaptive methods break the spell?



SPLASH OF A MILK DROP



Chelton et al. (2004), *Science*

The Gerris solver

- A framework to solve partial differential equations on quad/octree finite-volume meshes
- Basic blocks: Advection–diffusion, Poisson, Helmoltz solvers
- Combination: Euler, Stokes, Navier–Stokes, Saint-Venant etc...
- Volume-Of-Fluid interface tracking, accurate surface tension
- Free Software (General Public License): `http://gfs.sf.net`
- Parallel with dynamic load-balancing
- 365 registered users: universities, research institutes, industry, independent etc...

- Popinet (2003, 2009), *Journal of Computational Physics*, Popinet & Rickard (2007), Rickard, O’Callaghan & Popinet (2010) *Ocean Modelling*

Gerris' user interface

Different levels

Beginner "Numerical experiment" already set up: tune the parameters

Intermediate Set up a numerical experiment from scratch

Advanced Change the system of equations (in the parameter file)

Programmer Change the source code

Several generic but customisable solvers

Incompressible Euler Stokes, Navier-Stokes, variable-density, Boussinesq

Saint-Venant Friction, Coriolis, moving bathymetry

Electrohydrodynamics Incompressible Euler + electrostatic

The Navier–Stokes equations

- Mass conservation (Euler, 1772)

$$\frac{\mathrm{D}\int\rho}{\mathrm{D}t} = 0$$

- Momentum conservation (Euler, 1772)

$$\frac{\mathrm{D}\int\rho\mathbf{U}}{\mathrm{D}t} = \int\nabla p + \int \text{sources/sinks} + \int \text{dissipation?}$$

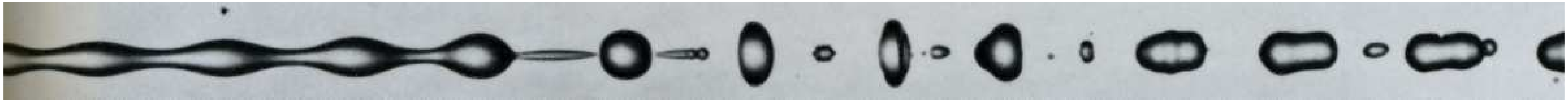
- Rheological properties of liquids (Navier, 1835 & Stokes, 1845)

$$\text{dissipation} \propto \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\mathbf{U}\times\boldsymbol{\nabla}^{\mathrm{T}}\mathbf{U})$$

Applicable to fluids in thermodynamic equilibrium from micrometres scales

(Stokes flows) to thousands of kilometres (geophysical fluid dynamics)

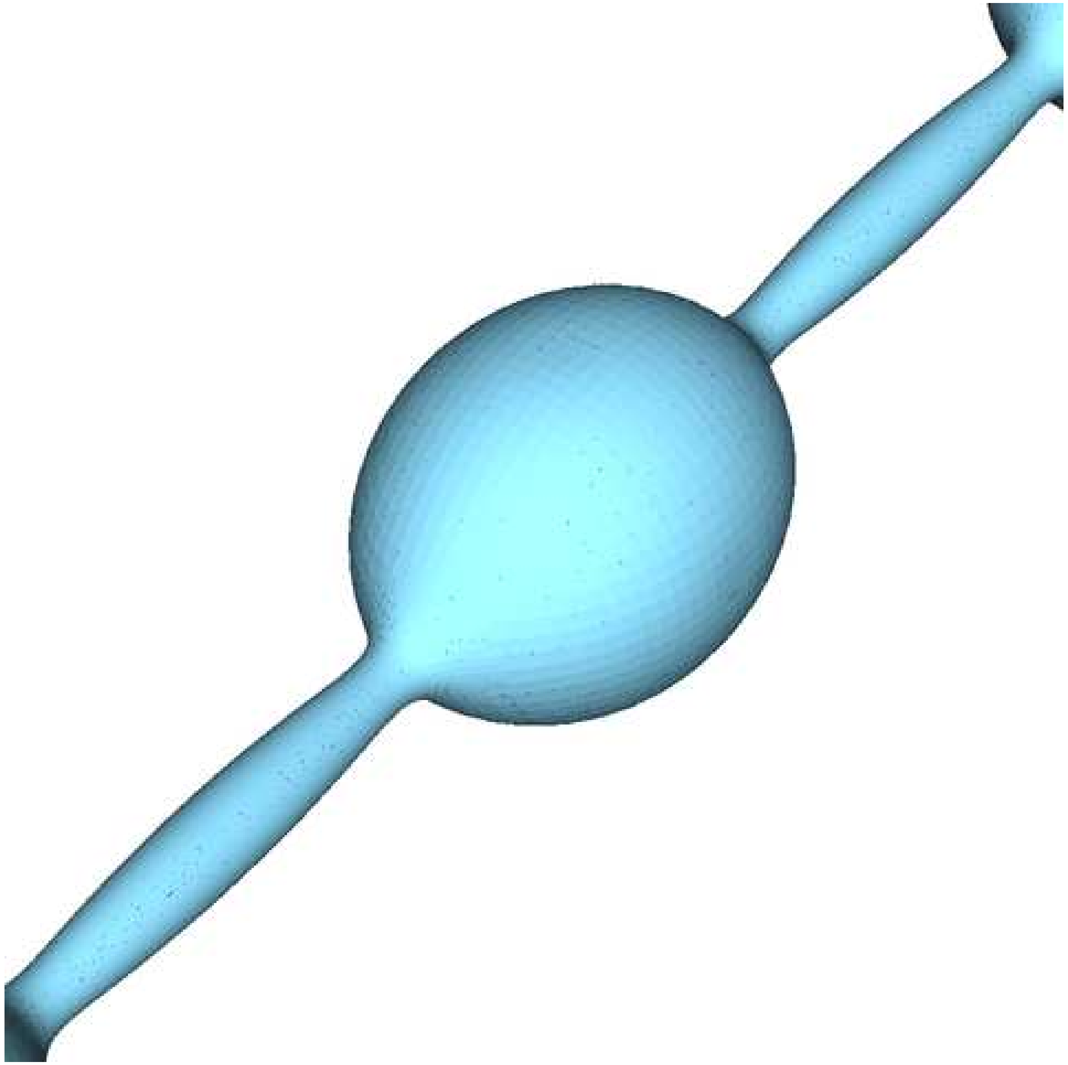
Savart–Plateau–Rayleigh instability



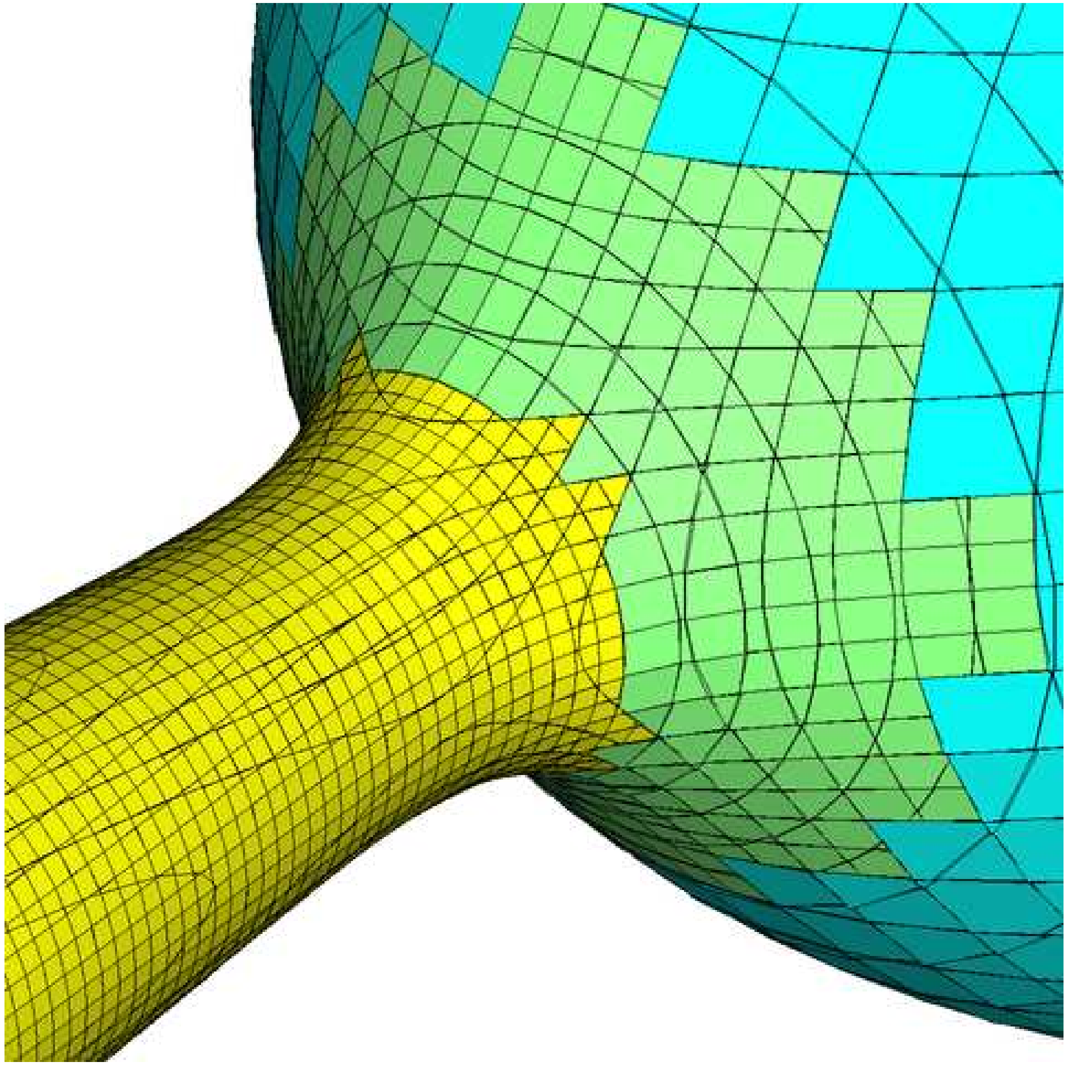
from Jameson (1971)

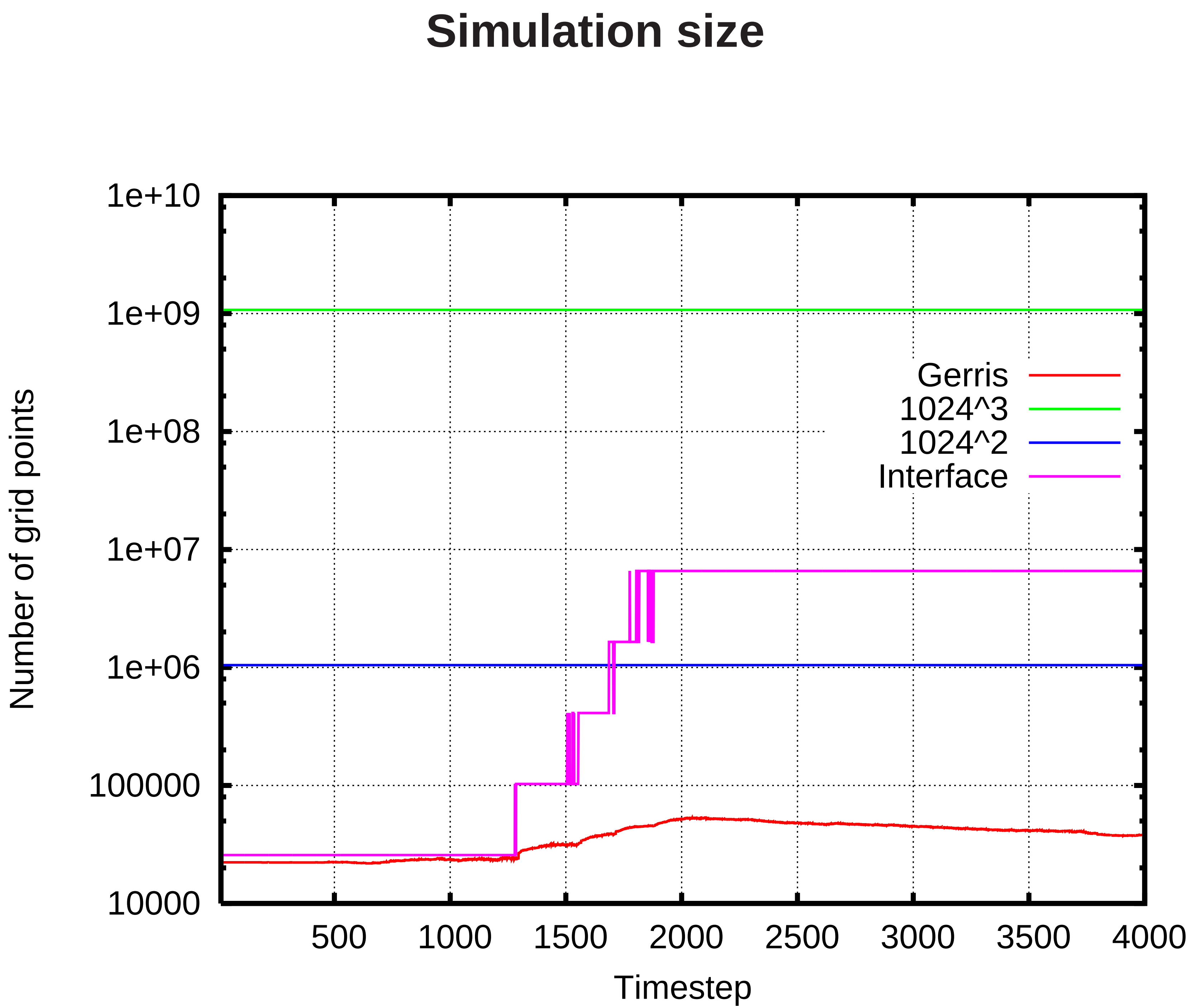
- Air/water physical parameters for a 0.5 mm water jet
- $x = 2\pi r_0/\lambda = 0.7$, $\mathrm{Re} = (r_0\sigma/\rho\nu^2)^{1/2} = 100$
- Mesh size controlled by $\kappa_{\mathrm{max}}\Delta < 0.2$ down to $\Delta = 2^{-10} \approx 10^{-3}$
- Full 3D incompressible Navier–Stokes simulation
- 12000 timesteps $\approx$ 2 hours CPU time (Desktop PC)

Evolution of the interface



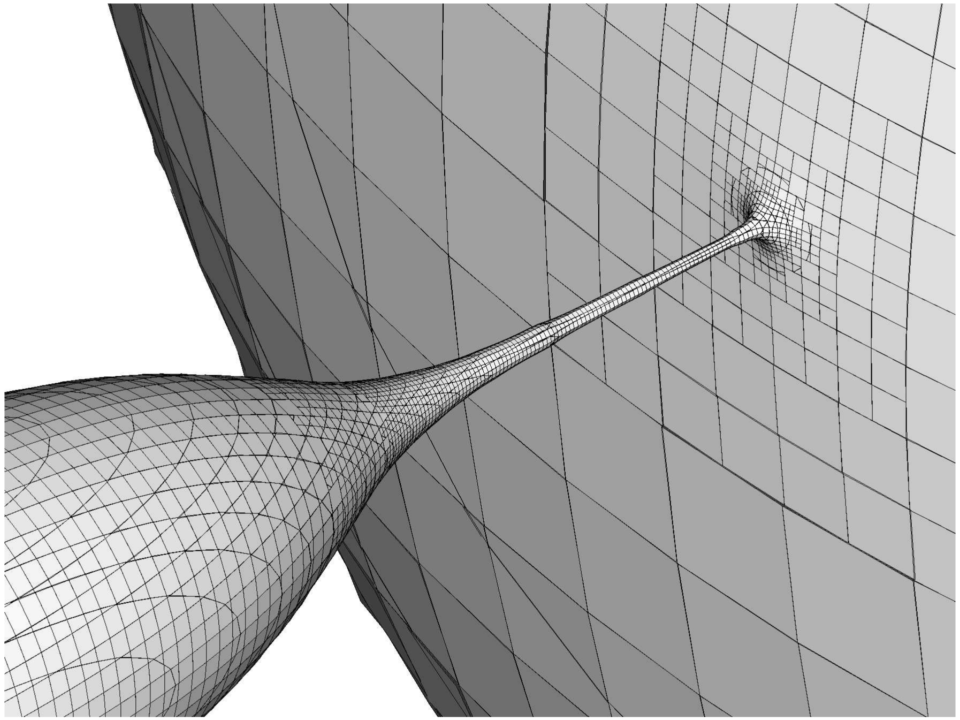
Close-up

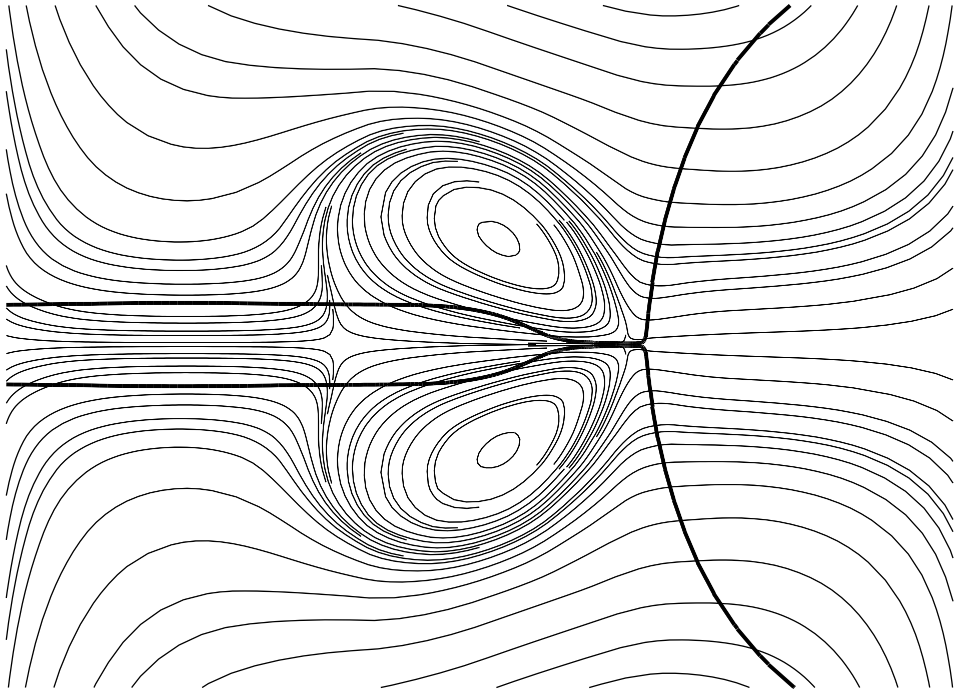




Simulation size
1e+10
1e+09
1e+08
1e+07
1e+06
100000
10000
500
1000
1500
2000
2500
3000
3500
4000
Number of grid points
Timestep
Gerris
1024^3
1024^2
Interface

Very viscous fluid ($La \approx 24$)

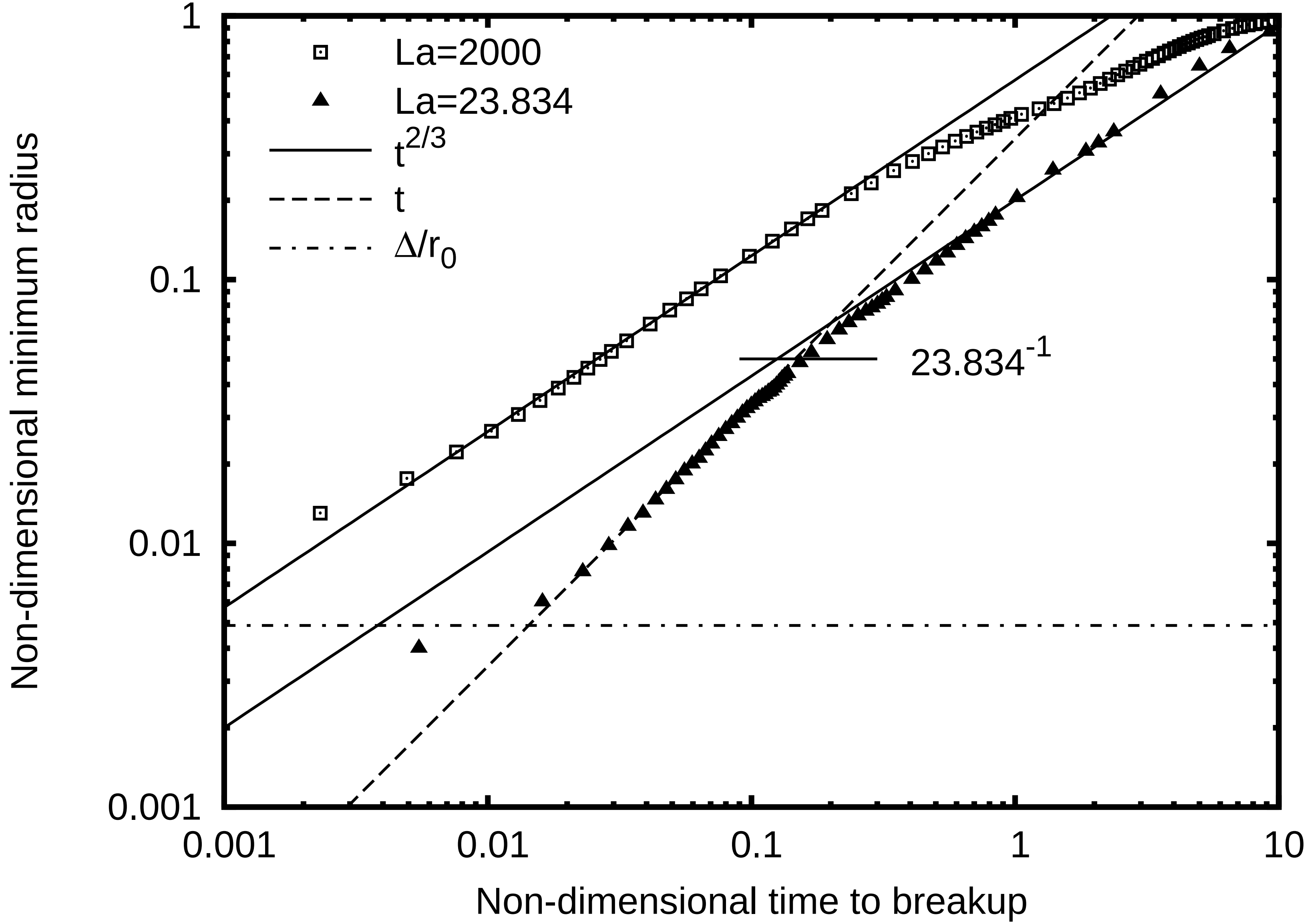




Thread diameter comparable to capillary length scale

$$l_\nu = \mu^2/\rho\sigma$$

Self-similar singularity close to breakup

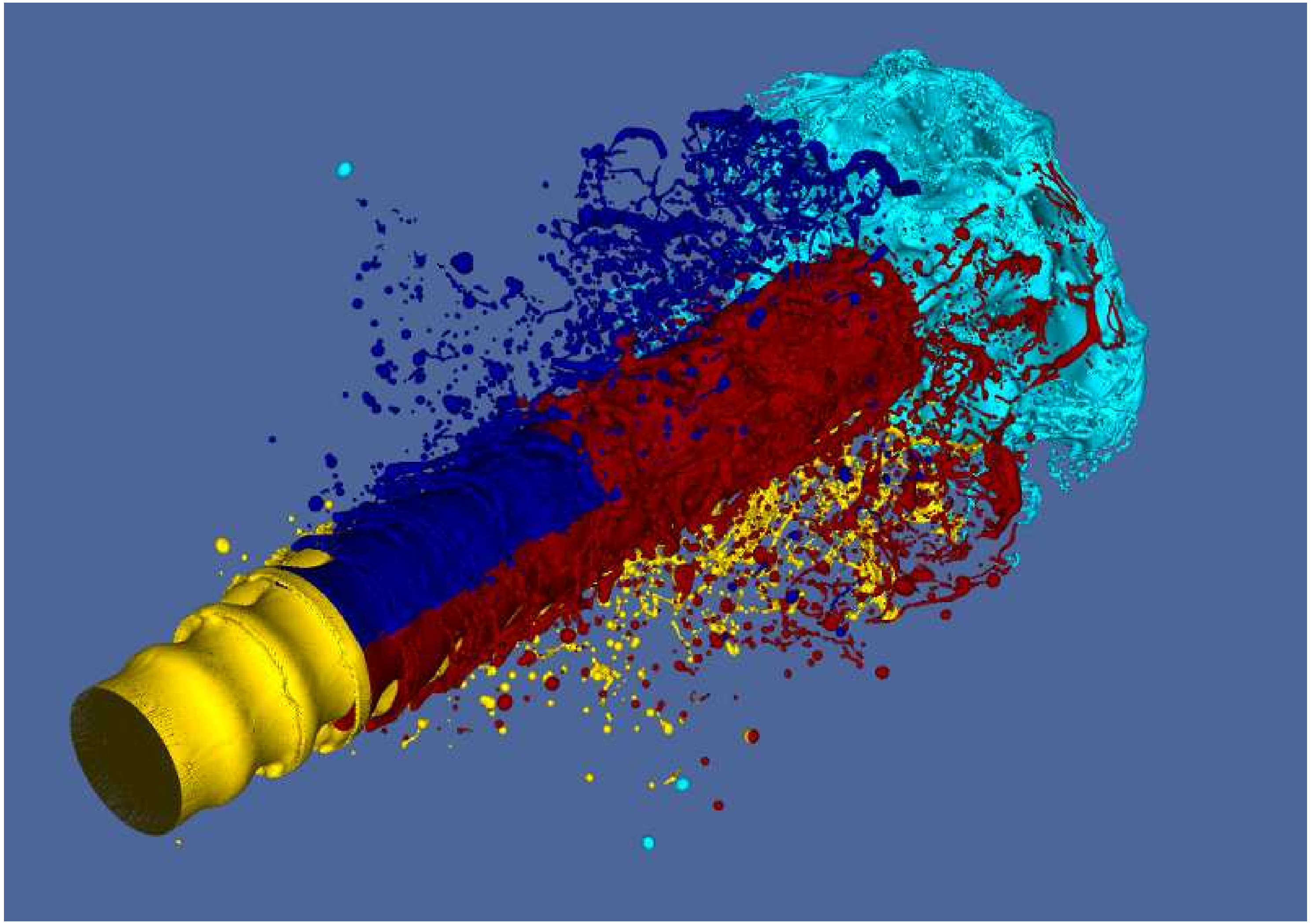


Gerris parameter file

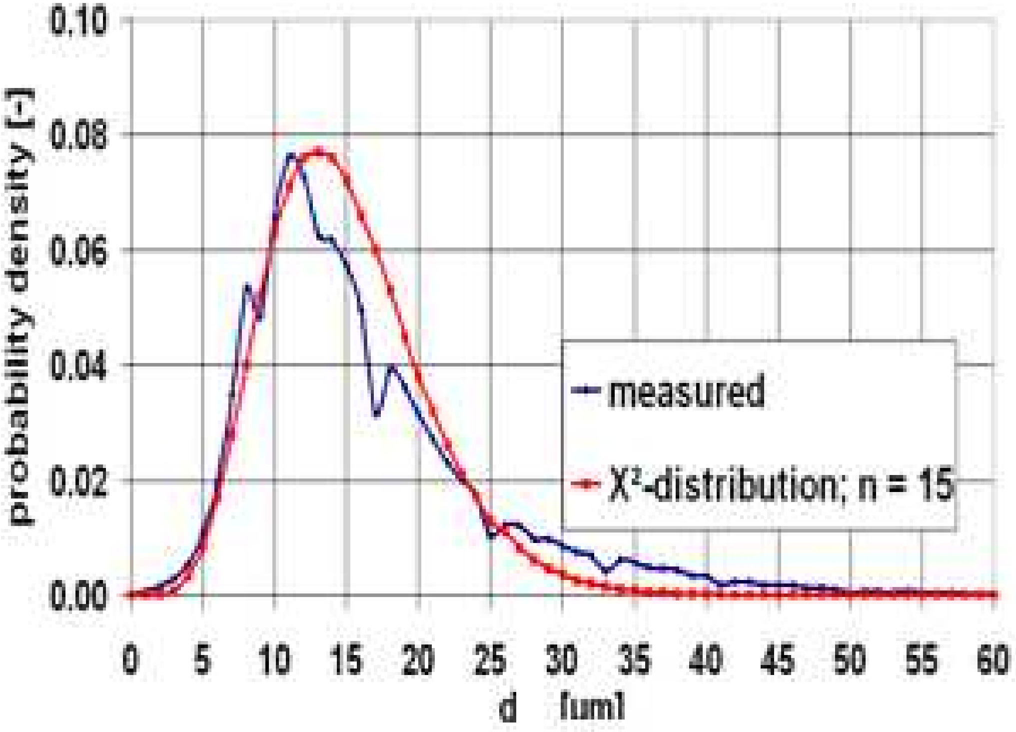
```
    VariableTracerVOF T
    # Filter the volume fraction for smoother density transition for
    # high density ratios: this helps the Poisson solver
    VariableFiltered T1 T 1
    PhysicalParams { alpha = 1./RHO(T1) }

    # We need Kmax as well as K(mean) for adaptivity
    VariableCurvature K T Kmax
    SourceTension T 1 K
    VariablePosition Y T y
    VariablePosition Z T z
#    SourceViscosityExplicit 1e-2*MUR(T1)
    SourceViscosity 1e-2*MUR(T1)

    # Initial deformed tube (only a quarter of it)
    InitFraction {} T ({
                 x -= 0.5;
                 y += 0.5; z += 0.5;
                 double r = RADIUS*(1. + EPSILON*cos(M_PI*x));
                 return r*r - y*y - z*z;
                 })
```

Atomisation of a pulsed diesel jet

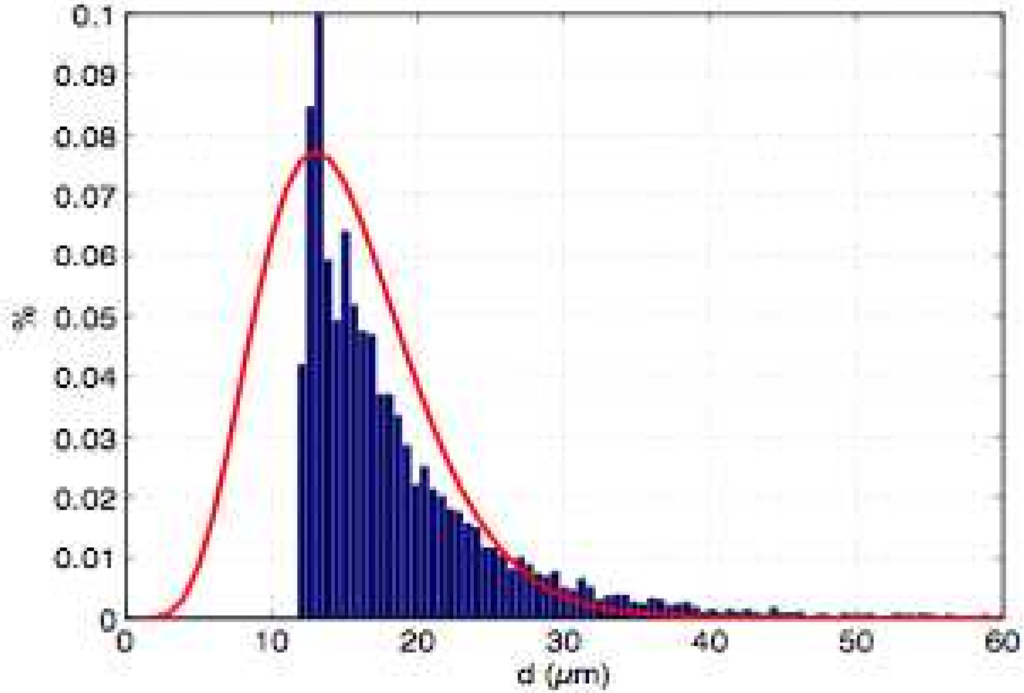
Parallel load-balancing



Droplet size distribution

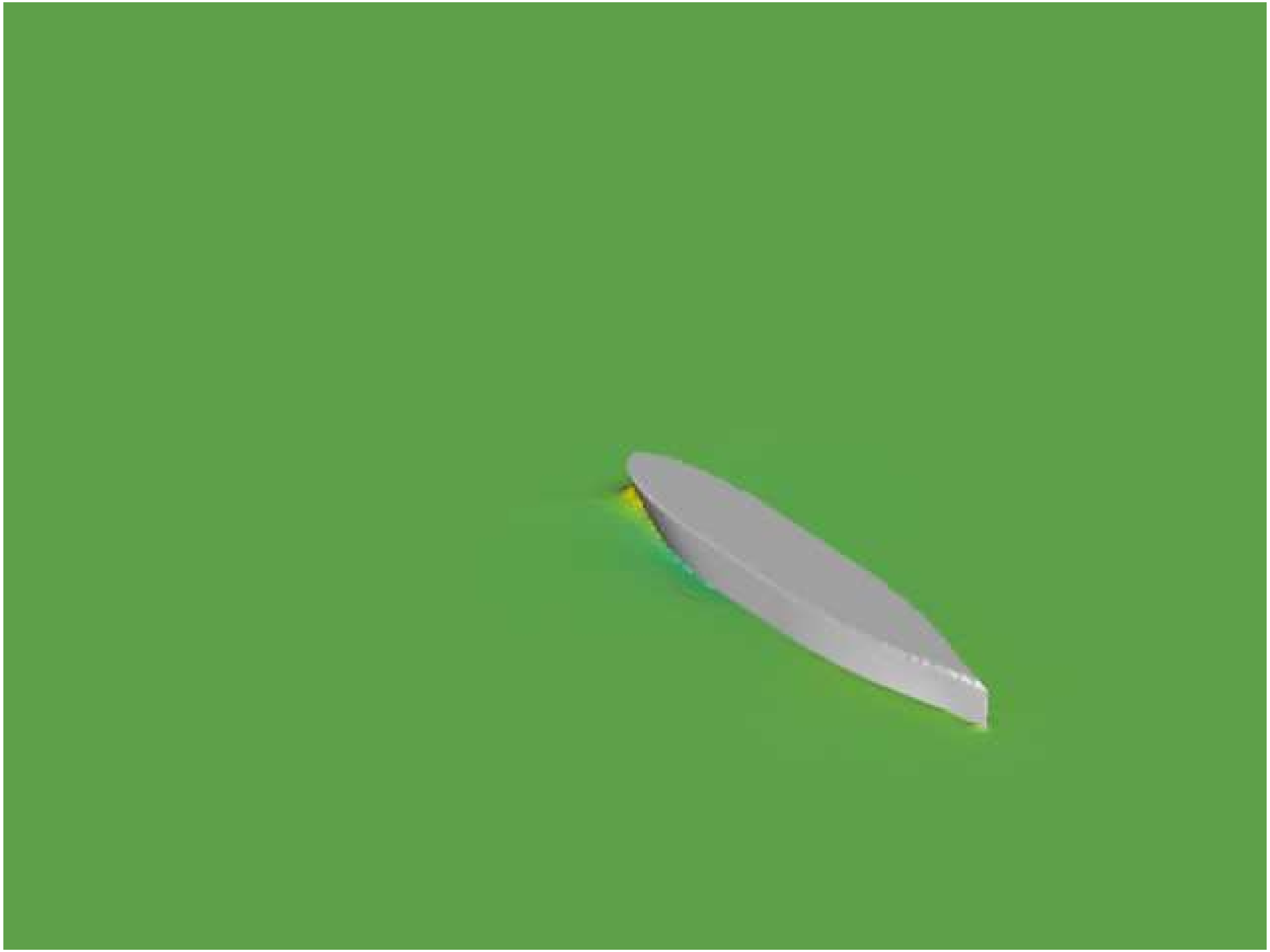


Experimental

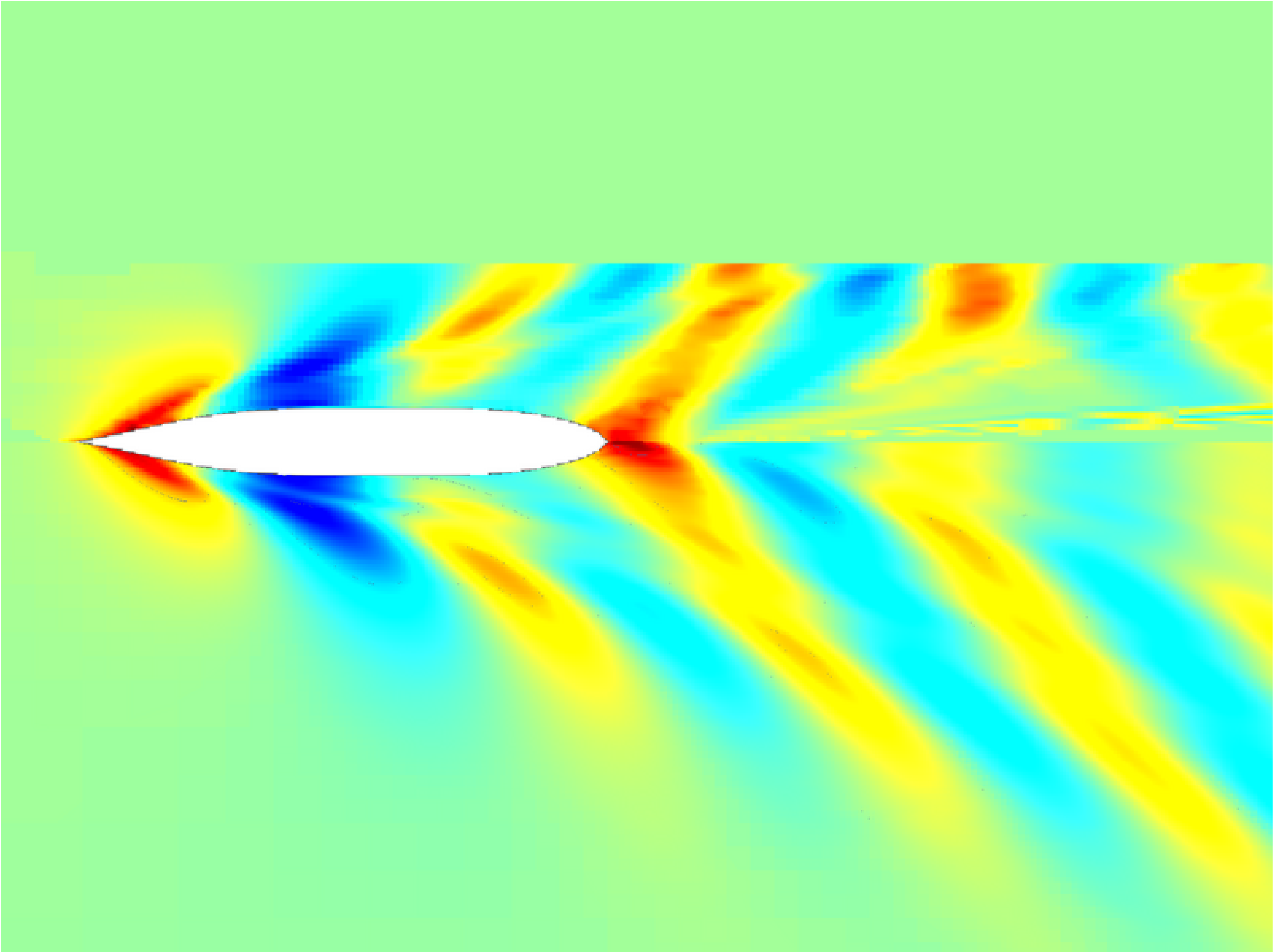


Numerical

Ship waves



Comparison with wave tank measurements



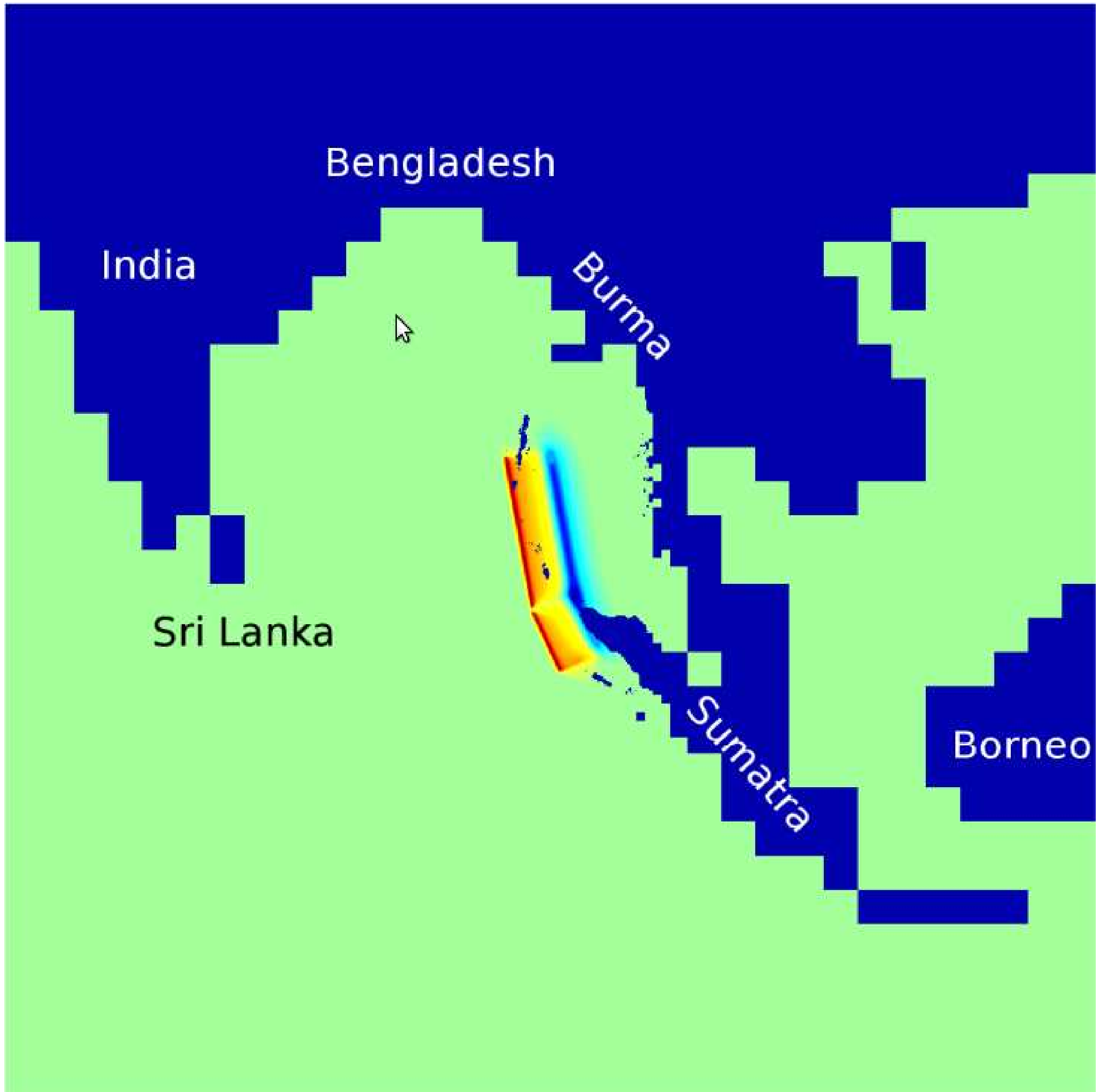
From small-scale Navier–Stokes to large-scale geophysical flows

- Long-wave approximation: the Saint-Venant equations
- Adhémar Jean Claude Barré de Saint-Venant, 1871. *Théorie du mouvement non-permanent des eaux, avec application au crues de rivières et à l'introduction des marées dans leur lit*
- (he also derived and published the "Navier–Stokes" equations in 1843, two years before Stokes)
- Depth averaging of the Navier–Stokes equations *or* mass/momentum balance in vertical slices
- No vertical structure implies parameterisation of vertical processes (e.g. friction, wave breaking etc...)

- The 2D Euler equations for compressible gases are identical (with the right equation of state) to the Saint-Venant equations.

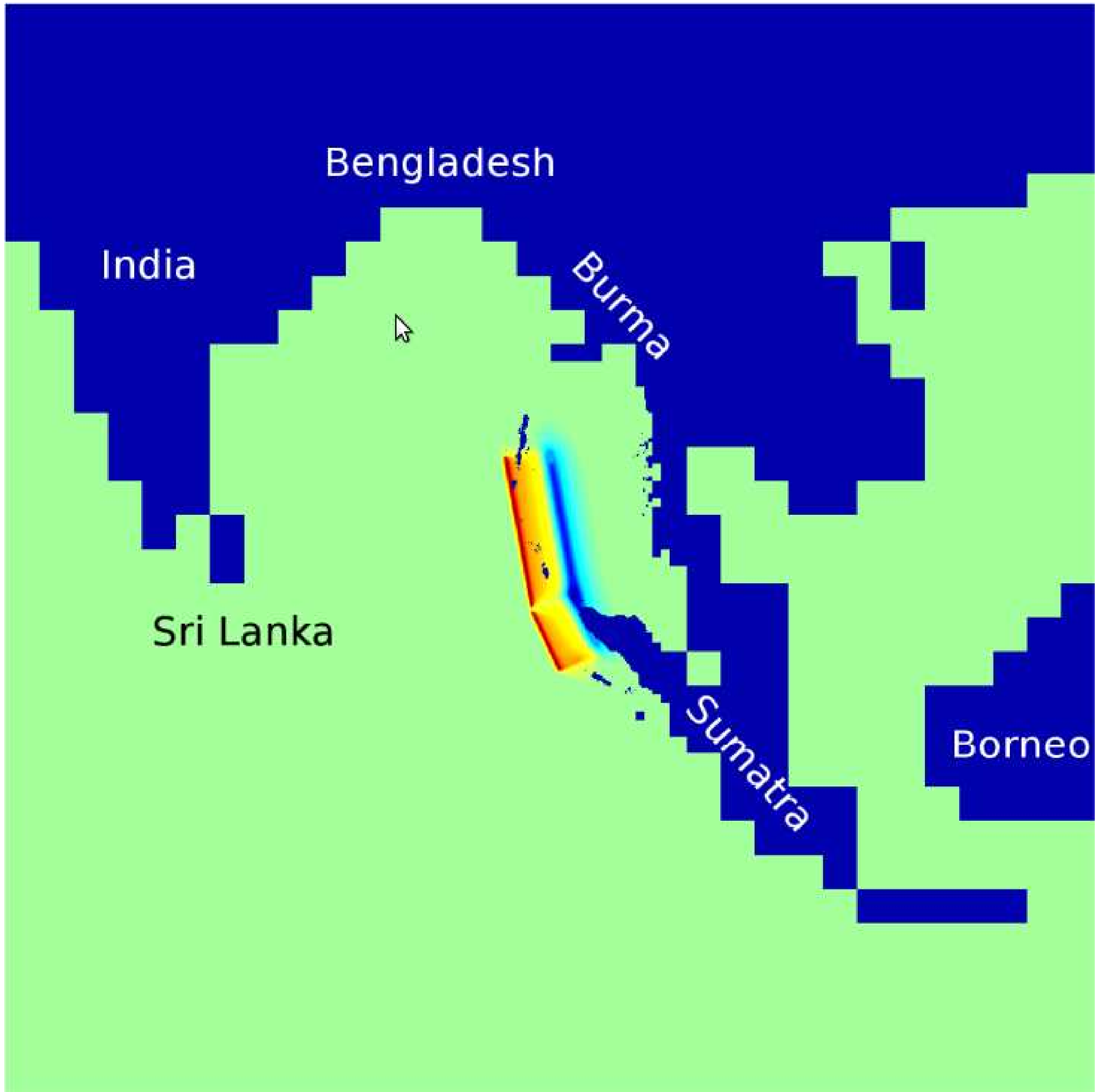
Flux	$\Longleftrightarrow$	Momentum
Depth	$\Longleftrightarrow$	Density
Hydraulic jumps	$\Longleftrightarrow$	Shocks
Wetting/drying	$\Longleftrightarrow$	Vacuum transitions

2004 Indian ocean tsunami



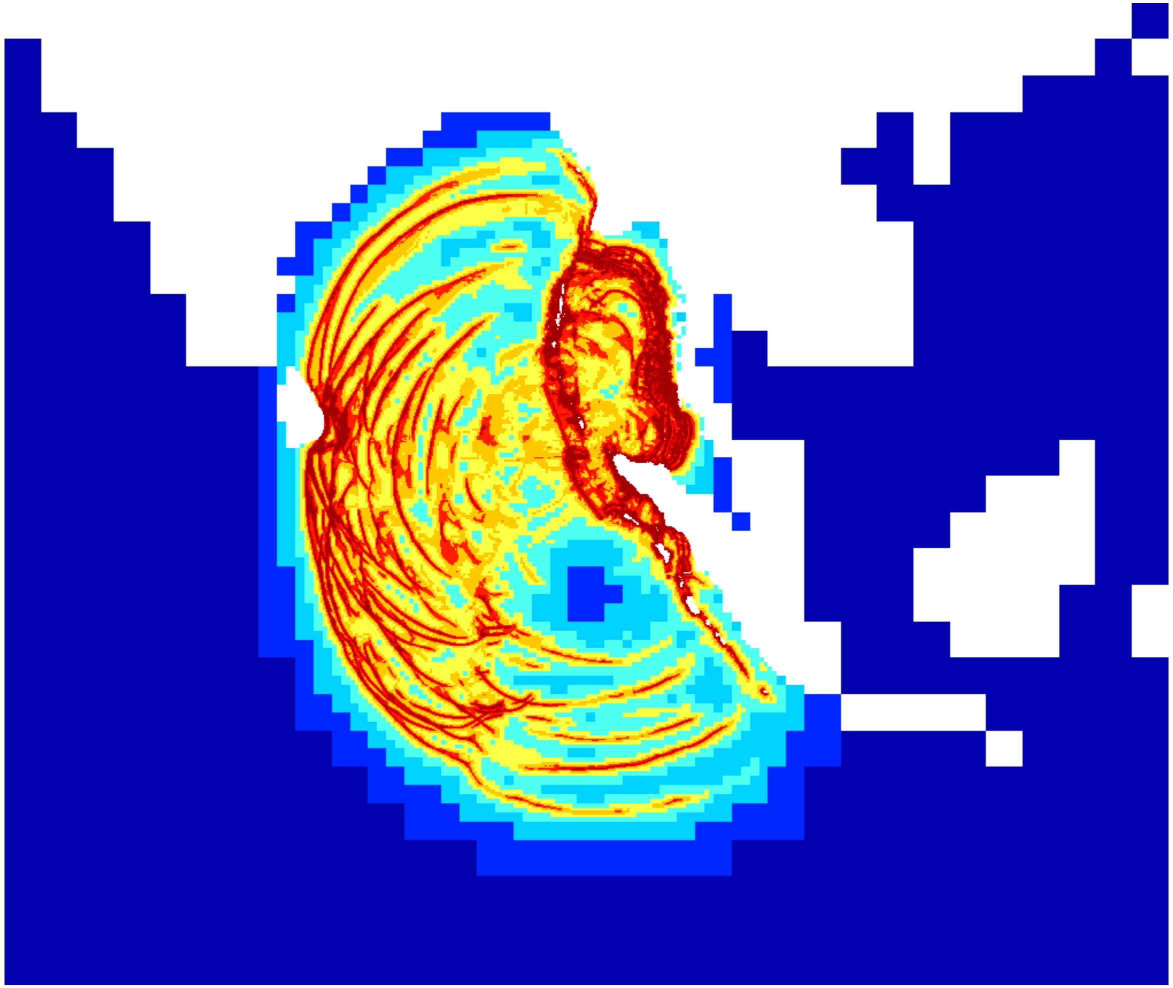
Staggered fault displacement model (5 segments)

2004 Indian ocean tsunami



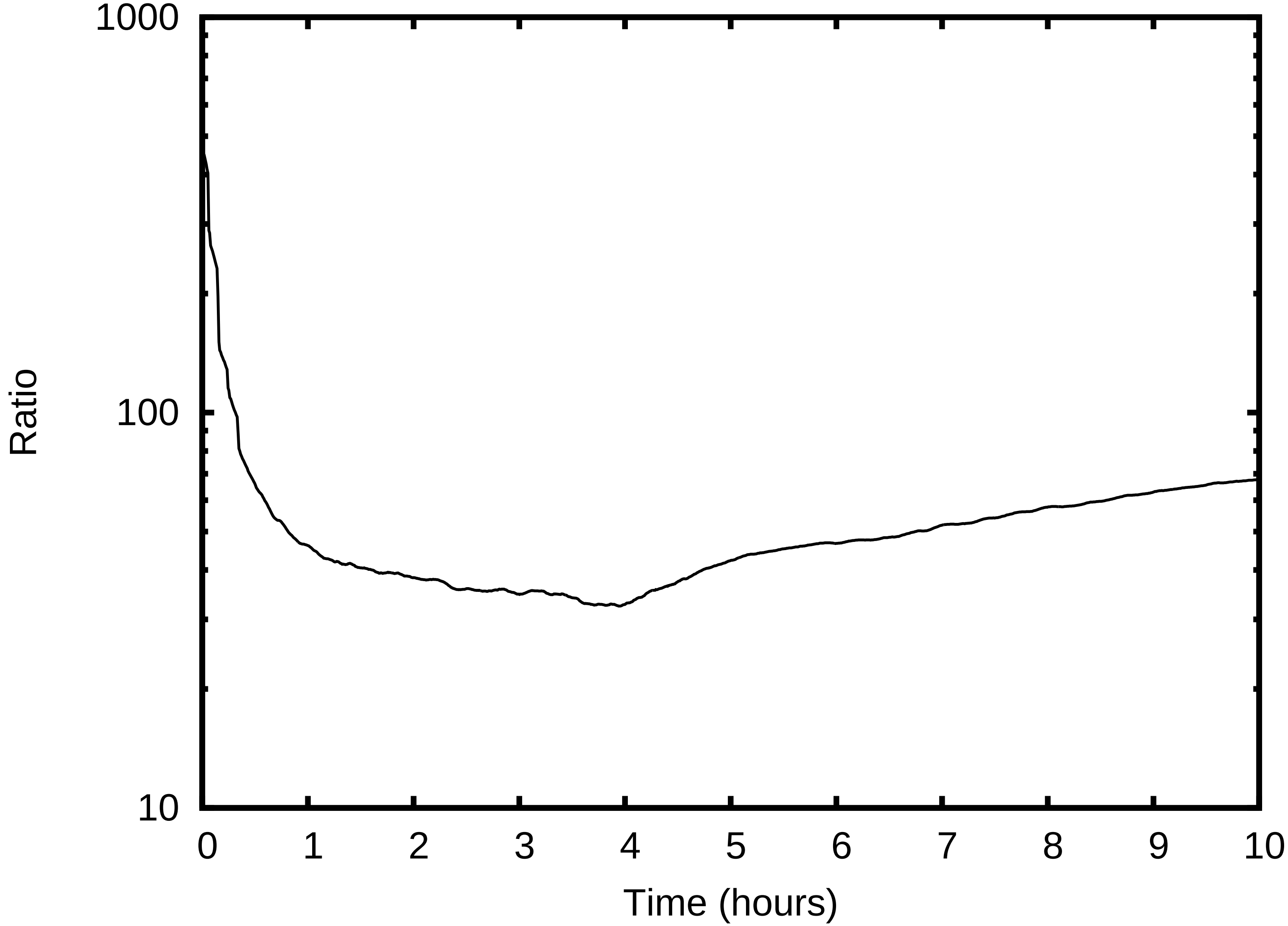
1 km $\leq$ Spatial resolution $\leq$ 150 km

Adaptivity

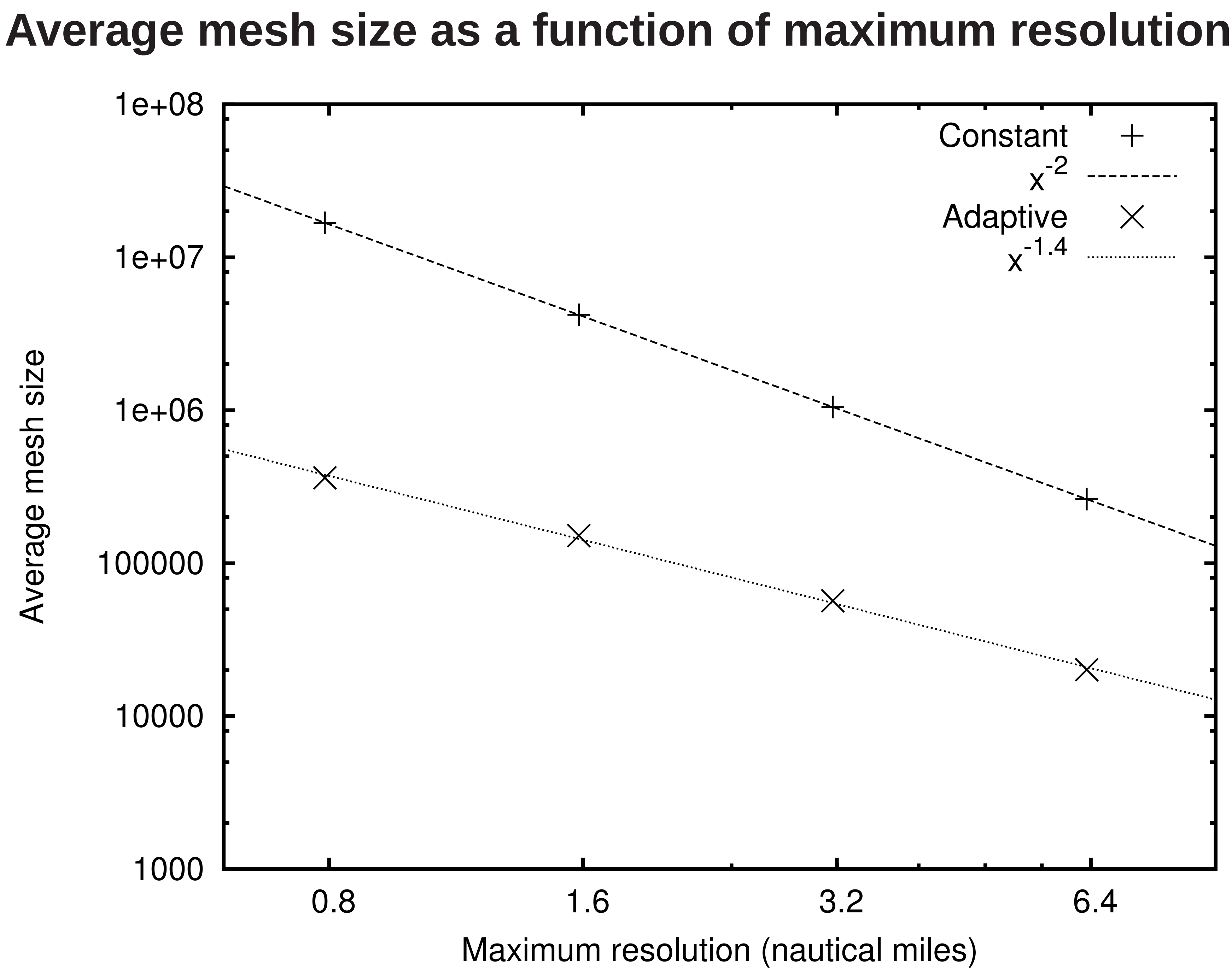


Truncation error of the wave height $<$ 5 cm

Simulation size compared to Cartesian discretisation



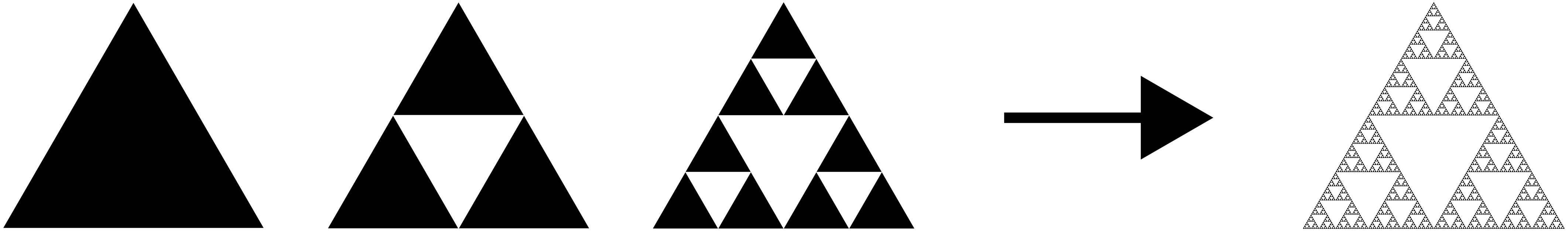
Of course this curve is resolution-dependent



Average mesh size as a function of maximum resolution
1e+08
1e+07
1e+06
100000
10000
1000
0.8
1.6
3.2
6.4
Average mesh size
Maximum resolution (nautical miles)
Constant
x-2
Adaptive
x-1.4

Connection with fractal dimension

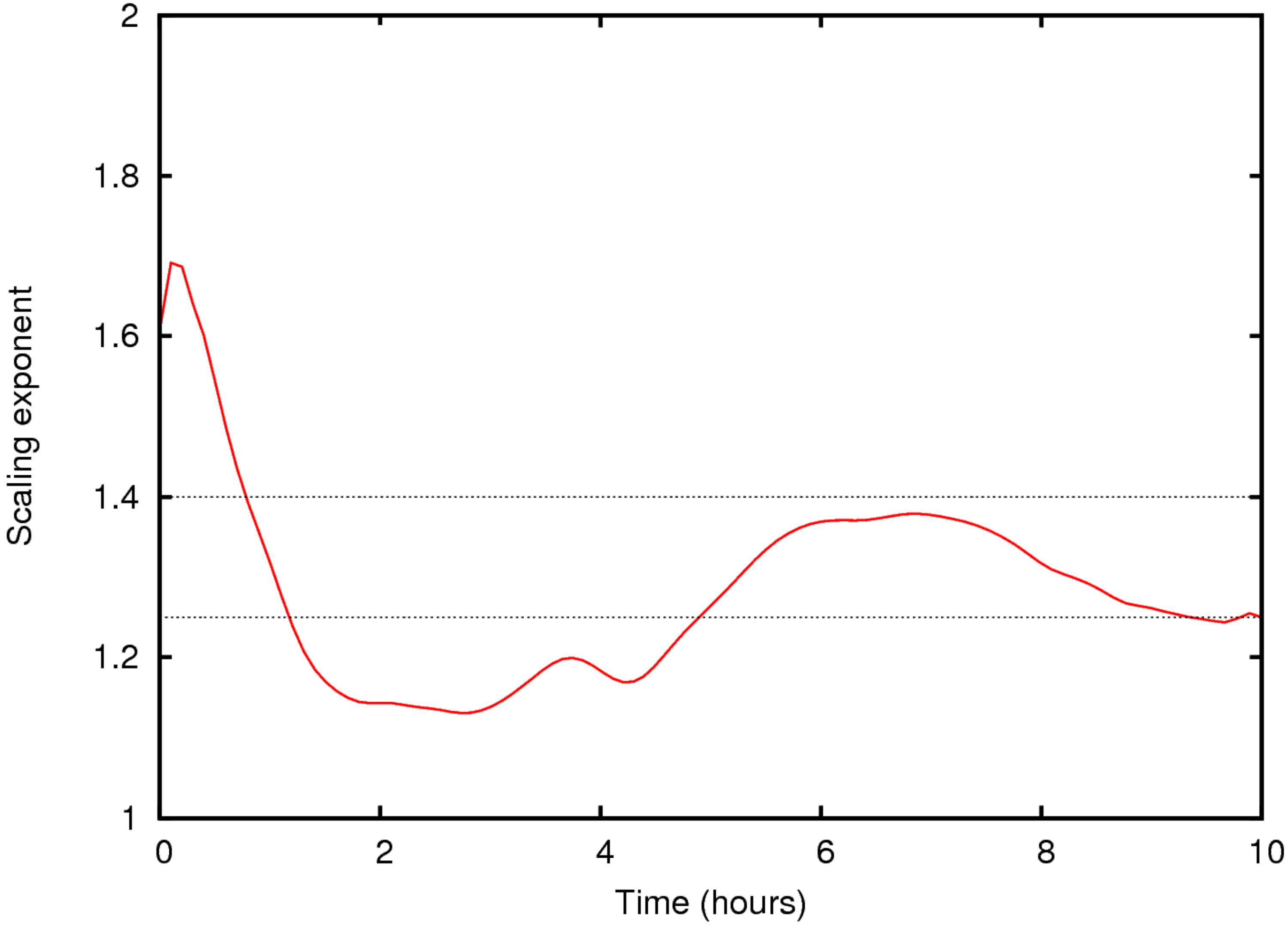
Classical example: the Sierpinski triangle



has a fractal (Minkowski–Bouligand or “box-counting” or “information”) dimension of ≈ 1.6.

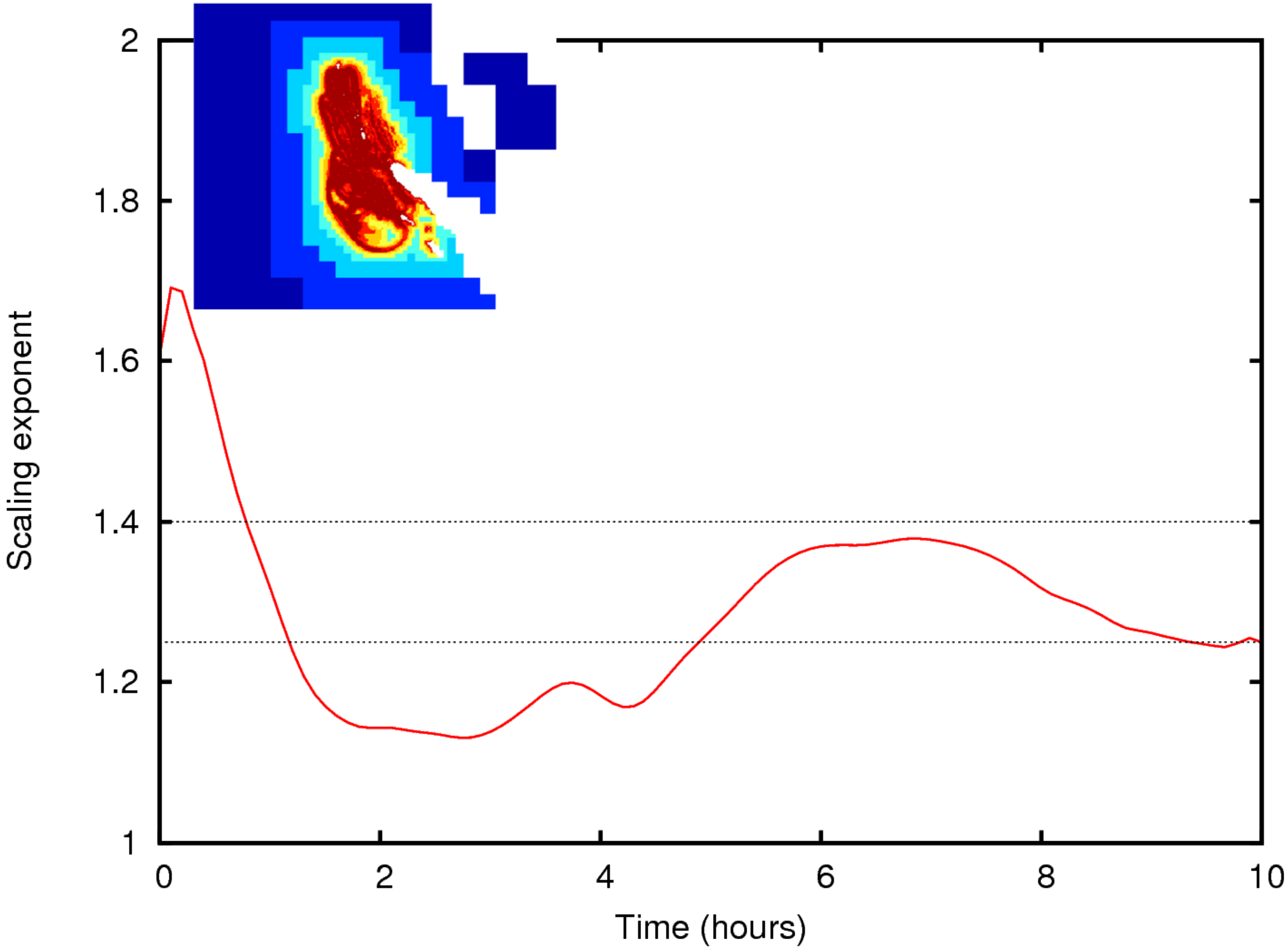
In other words, the cost of describing such an object using quadtrees would scale as $\Delta^{-1.6}$ not Δ^{-2}.

Evolution of the scaling exponent with time



Mandelbrot, *How long is the coast of Britain?*, Science, 1967

Evolution of the scaling exponent with time



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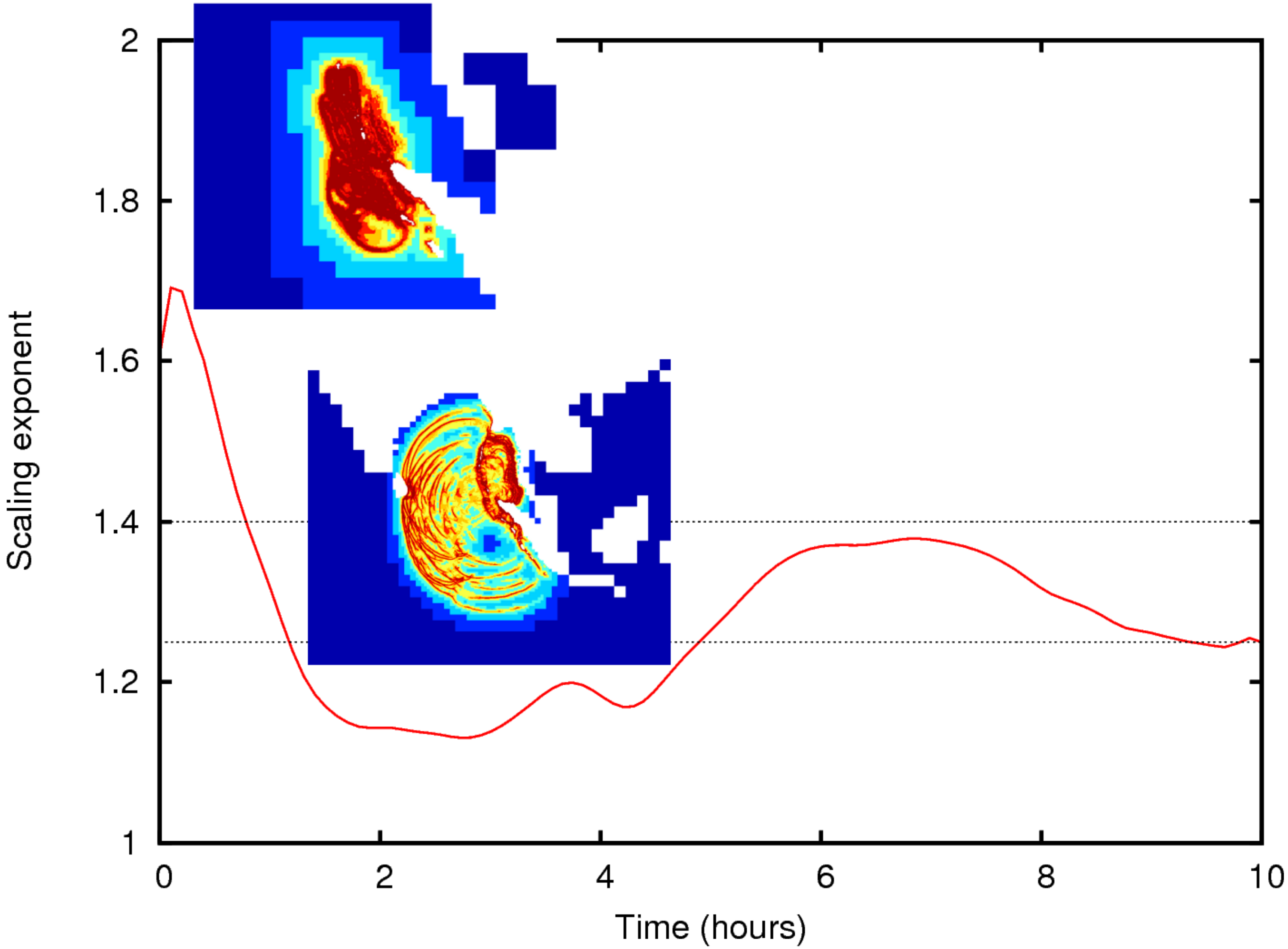

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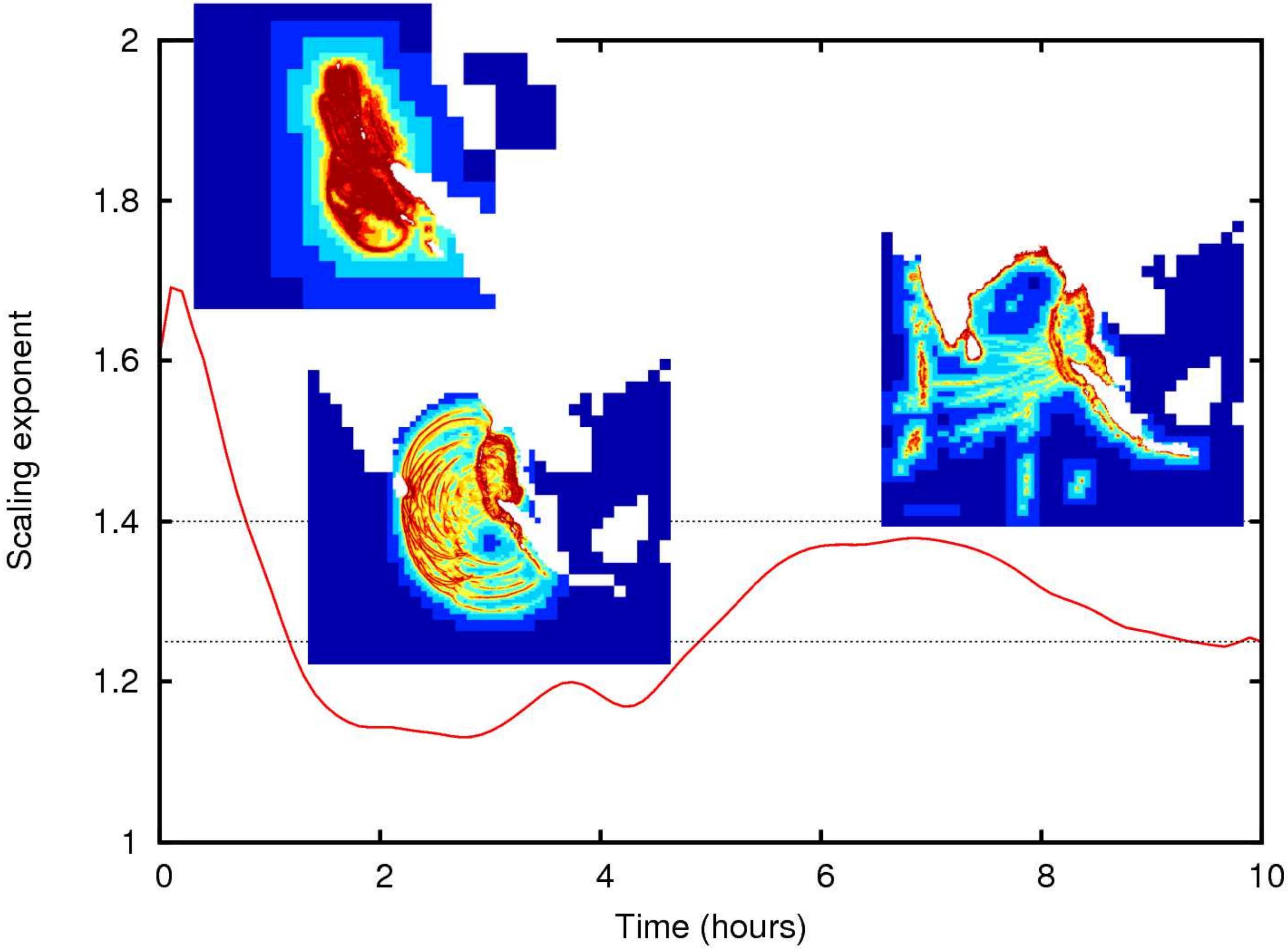

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Conclusions

- Adaptivity changes the scaling of computing costs: $C\Delta^{-d}$, d is now smaller than the number of dimensions

- This leads to orders-of-magnitude savings $\Longrightarrow$ new possibilities

- The user interface has proven to be well-suited for fast testing of “research ideas”

- “Reproducible research” should be a “must” for numerical experiments

- Many other applications: non-Newtonian rheologies, flow solutions on manifolds, spectral wave models etc...